

An example of a Fraïssé class without a Katětov functor

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Abstract

We disprove a conjecture from [2] by showing the existence of a Fraïssé class \mathcal{C} which does not admit a Katětov functor. On the other hand, we show that the automorphism group of the Fraïssé limit of \mathcal{C} is universal, as it happens in the presence of a Katětov functor.

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1 Introduction

Let us recall some basic facts. A Fraïssé class \mathcal{C} is a countable class of finitely generated structures for some countable language \mathcal{L} together with embeddings which satisfy joint embedding property, amalgamation property, hereditary property. Let us denote by $\sigma\mathcal{C}$ the class of all countably generated structures that are colimits of countable chains from \mathcal{C} . Obviously $\mathcal{C} \subseteq \sigma\mathcal{C}$. Fraïssé theorem says that there exists a unique homogeneous structure $\text{Flim}(\mathcal{C}) \in \sigma\mathcal{C}$, called the *Fraïssé limit* of \mathcal{C} , which is universal for $\sigma\mathcal{C}$. This means that every isomorphism between finitely generated substructures of $\text{Flim}(\mathcal{C})$ extends to an automorphism of $\text{Flim}(\mathcal{C})$, and every structure from \mathcal{C} embeds into $\text{Flim}(\mathcal{C})$. For more information see, e.g. [1].

In [2] the authors define a notion of a Katětov functor. Existence of such a functor leads to a uniform way of obtaining the Fraïssé limit and to prove, for example, that $\text{Aut}(\text{Flim}(\mathcal{C}))$ is universal for all $\text{Aut}(X)$ where $X \in \sigma\mathcal{C}$. Let us recall the definition.

Definition 1. We say that $K : \mathcal{C} \rightarrow \sigma\mathcal{C}$ is a Katětov functor if

- K is a covariant functor,
- there is a natural transformation $\{\eta_x\}_{x \in \mathcal{C}}$ between $\text{id}_{\mathcal{C}}$ and K ,

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & K(x) \\ f \downarrow & & \downarrow K(f) \\ y & \xrightarrow{\eta_y} & K(y) \end{array}$$

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- every one point extension $y \in \mathcal{C}$ of $x \in \mathcal{C}$ is realized in $K(x)$ over $\eta_x(x)$.

$$\begin{array}{ccc} x & \xrightarrow{\quad} & K(x) \\ \downarrow & \nearrow & \\ y & & \end{array}$$

Every Katětov functor K naturally extends to a functor $K: \sigma\mathcal{C} \rightarrow \sigma\mathcal{C}$ with similar properties, i.e., there is a natural transformation between $id_{\sigma\mathcal{C}}$ and K , every one point extension $y \in \mathcal{C}$ of $x \in \mathcal{C}$ where $x \subseteq X \in \sigma\mathcal{C}$ is realized in $K(X)$ over $\eta_X(x)$. Moreover, iterating the Katětov functor ω —many times leads to another Katětov functor $K^\omega: \sigma\mathcal{C} \rightarrow \sigma\mathcal{C}$ such that for all $X \in \sigma\mathcal{C}$ it holds that $K(X) = \text{Flim}(\mathcal{C})$. Easy observation (using functoriality) then gives us that $\text{Aut}(X)$ embeds into $\text{Aut}(\text{Flim}(\mathcal{C}))$ and $\text{Emb}(X, Y)$ embeds into $\text{Emb}(\text{Flim}(\mathcal{C}))$ i.e. $K^\omega: \sigma\mathcal{C} \rightarrow \{\text{Flim}(\mathcal{C})\}$ is a faithful functor. See [2] for more information. It was an open problem whether there is a Fraïssé class without a Katětov functor. In this note we prove that the class of linearly ordered finite sets with colorings of pairs by countably many colors without monochromatic triangles is a Fraïssé class without a Katětov functor, however it admits a faithful functor as above.

2 The construction

Let us fix some Fraïssé class \mathcal{C} and denote its closure on colimits of countable chains by $\sigma\mathcal{C}$. When we have a Katětov functor $K: \sigma\mathcal{C} \rightarrow \sigma\mathcal{C}$ we will always assume that $K(X)$ is an extension of X i.e. η_X is inclusion. Let us denote a one point extension $y \in \mathcal{C}$ of $x \in \mathcal{C}$ by $y = \langle x, t \rangle$ (meaning that t is a single generator). Similarly, $\langle X, t \rangle$ will denote a one point extension of $X \in \sigma\mathcal{C}$, namely, a structure in $\sigma\mathcal{C}$ generated by $X \cup \{t\}$.

Definition 2. Let $x, y \in \mathcal{C}$ such that $x \subseteq y$ and $X, C \in \sigma\mathcal{C}$. Consider the following commutative diagram.

$$\begin{array}{ccc} y & \xrightarrow{e_2} & C \\ \uparrow i & & \uparrow f \\ x & \xrightarrow{e_1} & X \end{array}$$

Then we say that C is a *homogeneous extension* of X over $x \subseteq y$ if for all $\alpha \in \text{Aut}(X)$ such that $\alpha \upharpoonright x = id_x$, there is $\beta \in \text{Aut}(C)$ such that $\beta \upharpoonright y = id_y$ and the following diagram commutes.

$$\begin{array}{ccccc} y & \xrightarrow{e_2} & C & \xrightarrow{\beta} & C \\ \uparrow i & & \uparrow f & & \uparrow f \\ x & \xrightarrow{e_1} & X & \xrightarrow{\alpha} & X \end{array}$$

The most important case is when C is generated by $X \cup y$ and y is generated by one element over x . In that case we in fact deal with extending one point extension $\langle x, t \rangle$ to a homogeneous one point extension $\langle X, t \rangle$.

Proposition 1. Assume that there is a Katětov functor K for \mathcal{C} . Then for all $x \in \mathcal{C}$, all one point extensions $\langle x, t \rangle \in \mathcal{C}$ and all embeddings $e: x \rightarrow \text{Flim}(\mathcal{C})$ there exists a homogeneous one point extension $\langle \text{Flim}(\mathcal{C}), t \rangle$ of $\text{Flim}(\mathcal{C})$ over $x \subseteq \langle x, t \rangle$.

Proof. Assume we have the following commutative diagram, where α is an automorphism.

$$\begin{array}{ccc} x & \xrightarrow{e} & \text{Flim}(\mathcal{C}) \\ id \uparrow & & \uparrow \alpha \\ x & \xrightarrow{e} & \text{Flim}(\mathcal{C}) \end{array}$$

This diagram can be moved by our Katětov functor K to the following commutative diagram

$$\begin{array}{ccc} K(x) & \xrightarrow{K(e)} & K(\text{Flim}(\mathcal{C})) \\ id \uparrow & & \uparrow K(\alpha) \\ K(x) & \xrightarrow{K(e)} & K(\text{Flim}(\mathcal{C})) \end{array}$$

Where $K(\alpha)$ is again an automorphism, and these two diagrams are connected by the natural transformation η . Then $\langle x, t \rangle$ can be embedded to $K(x)$ by the definition of K , so we may assume that this embedding is inclusion and put $e(t) := K(e)(t)$. By commutativity we see that $K(\alpha)(e(t)) = e(t)$. So $K(\alpha)$ and $K(\alpha)^{-1}$ are invariant on $\langle \text{Flim}(\mathcal{C}), e(t) \rangle$ which is exactly what we needed to prove. \square

Another consequence of the existence of a Katětov functor is a nontrivial pair of embeddings $e: \text{Flim}(\mathcal{C}) \rightarrow \text{Flim}(\mathcal{C})$ and $E: \text{Aut}(\text{Flim}(\mathcal{C})) \rightarrow \text{Aut}(\text{Flim}(\mathcal{C}))$ such that the following diagram

$$\begin{array}{ccc} \text{Flim}(\mathcal{C}) & \xrightarrow{e} & \text{Flim}(\mathcal{C}) \\ \alpha \uparrow & & \uparrow E(\alpha) \\ \text{Flim}(\mathcal{C}) & \xrightarrow{e} & \text{Flim}(\mathcal{C}) \end{array}$$

commutes for all $\alpha \in \text{Aut}(\text{Flim}(\mathcal{C}))$. Assume that we have such a pair (e, E) , fix $x \subseteq \text{Flim}(\mathcal{C})$ and $t \in \text{Flim}(\mathcal{C}) \setminus e[\text{Flim}(\mathcal{C})]$. Define

$$\mathcal{G} := \{\alpha \in \text{Aut}(\text{Flim}(\mathcal{C})) : E(\alpha) \upharpoonright \langle x, t \rangle = id_{\langle x, t \rangle}\}.$$

This is an open subgroup of $\text{Aut}(\text{Flim}(\mathcal{C}))$ because it is an analytic subgroup (i.e. it has the Baire property) and it has countable index. So there is $x \subseteq x' \subseteq \text{Flim}(\mathcal{C})$ such that $\mathcal{H} := \{\alpha : \alpha \upharpoonright x' = id_{x'}\} \leq \mathcal{G}$ i.e. $\langle \text{Flim}(\mathcal{C}), t \rangle$ is a one point homogeneous extension of $\text{Flim}(\mathcal{C})$ over $\langle x', t \rangle$. Hence we may say that for a nontrivial pair of embeddings (e, E) and every one point extension $\langle x, t \rangle$ such that $x \subseteq \text{Flim}(\mathcal{C})$ and t is realized in $\text{Flim}(\mathcal{C})$ over $e[x]$ outside $e[\text{Flim}(\mathcal{C})]$ there are $x' \supseteq x$ and a homogeneous one point extension $\langle \text{Flim}(\mathcal{C}), t \rangle$ of $\text{Flim}(\mathcal{C})$ over $\langle x', t \rangle$. In particular, if (e, E) is nontrivial then there are nontrivial homogeneous one point extensions.

As it was mentioned in the introduction, iterating ω many times a fixed Katětov functor K leads to the functor K^ω such that $K^\omega(X) = \text{Flim}(\mathcal{C})$ for every $X \in \sigma\mathcal{C}$. In particular, $K^\omega : \sigma\mathcal{C} \rightarrow \{\text{Flim}(\mathcal{C})\}$ is a faithful functor.

Theorem 1. *There exists a Fraïssé class \mathcal{C} without a Katětov functor, yet with a faithful functor from $\sigma\mathcal{C}$ to $\{\text{Flim}(\mathcal{C})\}$.*

Proof. Let Q be a countable set. Let us define the class \mathcal{C} . An element of \mathcal{C} is a finite set x with a linear order and with a function $c_x : [x]^2 \rightarrow Q$ such that there are no monochromatic triangles. We will denote the coloring function c_x always by c , omitting the subscript x . It can be easily seen that \mathcal{C} is a Fraïssé class.

Let us first prove that there are no homogeneous one point extensions of $\text{Flim}(\mathcal{C})$. Let us fix $x \subseteq \text{Flim}(\mathcal{C})$ and any one point extension $\langle x, t \rangle \in \mathcal{C}$. Let us assume that there is a homogeneous one point extension $\langle \text{Flim}(\mathcal{C}), t \rangle$ of $\text{Flim}(\mathcal{C})$ over $\langle x, t \rangle$. Let us pick any $t_1 \in \text{Flim}(\mathcal{C})$ which realizes the same type over x as t . Let $q = c((t, t_1))$. Using the saturation of $\text{Flim}(\mathcal{C})$ we find an element $t_2 \in \text{Flim}(\mathcal{C})$ such that $c((t_2, t_1)) = q$ and the mapping $\alpha : \langle x, t_1 \rangle \rightarrow \langle x, t_2 \rangle$, defined by conditions $\alpha \upharpoonright x = \text{id}_x$ and $\alpha(t_1) = t_2$, is an isomorphism. Take any automorphism $\alpha_0 : \text{Flim}(\mathcal{C}) \rightarrow \text{Flim}(\mathcal{C})$ which extends α . From the definition of a homogeneous extension, α_0 extends to $\langle \text{Flim}(\mathcal{C}), t \rangle$ such that $\alpha_0(t) = t$. Then we have that $\{t, t_1, t_2\}$ is a monochromatic triangle, which is a contradiction.

From the arguments above it follows that not only there is no Katětov functor for \mathcal{C} but there is no nontrivial pair of embeddings (e, E) for $\text{Flim}(\mathcal{C})$ as well.

In order to prove that $\text{Emb}(\text{Flim}(\mathcal{C}))$ is universal for $\sigma\mathcal{C}$, let us define a suitable sequence of Fraïssé classes $\{\mathcal{C}_i\}_{i \leq \omega}$. First let us fix a sequence of countable sets $\{Q_i\}_{i \leq \omega}$ such that $|Q_{i+1} \setminus Q_i| = \omega$ and $Q_\omega = \bigcup_{i < \omega} Q_i$. The definition of \mathcal{C}_i is similar as of \mathcal{C} , the only difference is that the set of colors is Q_i . We have that $\sigma\mathcal{C}_i \subseteq \sigma\mathcal{C}_{i+1} \subseteq \sigma\mathcal{C}_\omega$. Now for each $i < \omega$ we will find a functor K_i such that

- $K_i : \mathcal{C}_i \rightarrow \sigma\mathcal{C}_{i+1}$,
- there is a natural transformation ν for inclusion and K_i , i.e., $\nu_x : x \rightarrow K_i(x)$ for $x \in \mathcal{C}_i$,
- for every $x \in \mathcal{C}_i$ every \mathcal{C}_i -type over x is realized in $\nu_x[x]$.

The functor K_i extends with all its properties (as in the case of Katětov functors) to $K_i : \sigma\mathcal{C}_i \rightarrow \sigma\mathcal{C}_{i+1}$. Once we have this, let us put

$$K_\omega = \dots \circ K_i \circ \dots \circ K_1 \circ K_0 : \sigma\mathcal{C}_0 \rightarrow \sigma\mathcal{C}_\omega.$$

The functor K_ω is correctly defined thanks to the natural transformations for the functors K_i , and there is a natural transformation from the inclusion $\sigma\mathcal{C}_0 \subseteq \sigma\mathcal{C}_\omega$ to K_ω . Moreover, we have that $K_\omega(X) = \text{Flim}(\mathcal{C}_\omega) \simeq \text{Flim}(\mathcal{C}_0)$, which proves our claim.

To finish the proof it is enough to describe K_0 . Let us put $K := K_0$, $\mathcal{C} := \mathcal{C}_0$, $Q := Q_0$, $\mathcal{D} := \mathcal{C}_1$, $P := Q_1$ and fix any $p \in P \setminus Q$. Let us define K on objects. Take any $x \in \mathcal{C}$. Assume that we have fixed a linear ordering $<_Q$ on Q isomorphic to the natural numbers. Denote by \mathcal{O}_x the set of all partial proper one-point extensions of x from \mathcal{C} . We will use the same letter for a type and for its realization. To every $\xi \in \mathcal{O}_x$ let $\text{supp}(\xi) \subseteq x$ denote the support of ξ , that is, the substructure of x to which ξ is added. Let $K(x) = x \cup \mathcal{O}_x$. Given an embedding $e : x \rightarrow y$, define $K(e)$ in the obvious way, namely, a partial type ξ is mapped to the corresponding partial type with support $e[\text{supp}(\xi)]$. Next we turn $K(x)$ to be an element of $\sigma\mathcal{D}$ in such a way that $K(e)$ remains an embedding of structures whenever e is an embedding.

Extend the coloring by putting $c(v, \xi) = p$ for $v \in x \setminus \text{supp}(\xi)$ and extend the ordering on ξ by declaring $\xi < v$ for $v \in x \setminus \text{supp}(\xi)$ whenever it is consistent with the ordering on $\text{supp}(\xi) \cup \{\xi\}$. In the next step we define a linear ordering on $K(x)$. Given $\xi \neq \psi \in \mathcal{O}_x$, let us define $\xi < \psi$ if one of the following conditions is satisfied.

- (1) there is $v \in x$ such that $\xi < v < \psi$,
- (2) condition (1) fails and $|\text{supp}(\xi)| < |\text{supp}(\psi)|$,
- (3) conditions (1), (2) fail and the biggest $v \in \text{supp}(\xi) \Delta \text{supp}(\psi)$ satisfies $v \in \text{supp}(\xi)$,
- (4) none of the above is satisfied and for the biggest $v \in \text{supp}(\xi) = \text{supp}(\psi)$ for which $c_\xi((v, \xi)) \neq c_\psi((v, \psi))$ we have that $c_\xi((v, \xi)) <_Q c_\psi((v, \psi))$.

It remains to define a coloring on $K(x)$ extending the coloring of x . In order to do this, let us define an equivalence relation on pairs of elements from \mathcal{O}_x . We say that $\{\xi_0 < \psi_0\} \sim \{\xi_1 < \psi_1\}$ if there is an isomorphism between $\text{supp}(\xi_i) \cup \text{supp}(\psi_i)$ where $i \in \{0, 1\}$ whose extension to $\{\xi_i < \psi_i\}$ remains an isomorphism. It is clear from the definition that if $\{\xi_0 < \psi_0\} \sim \{\xi_1 < \psi_1\}$ then $\text{supp}(\xi_0) \simeq \text{supp}(\xi_1)$ and $\text{supp}(\psi_0) \simeq \text{supp}(\psi_1)$. Furthermore, the isomorphisms are unique, because of the linear orderings. It follows immediately:

Claim 1. *For $f: x \rightarrow y$ we have that $\{\xi_0 < \psi_0\} \sim \{\xi_1 < \psi_1\}$ in \mathcal{O}_x iff $\{K(f)(\xi_0) < K(f)(\psi_0)\} \sim \{K(f)(\xi_1) < K(f)(\psi_1)\}$ in \mathcal{O}_y .*

Let us now color the equivalence classes by induction on the size of x . Assume that for all sets of size $< n$ we have already defined the coloring and take x such that $|x| = n$. For the equivalence class of a pair $\{\xi_0 < \psi_0\}$ use the color $r \in P$ if there is an embedding $f: y \rightarrow x$ such that $\{\xi_0 < \psi_0\}$ is in the image of $K(f)$ and their preimage is colored by r . This is well-defined by Claim 1. Color the remaining equivalence classes by different (not already used) colors so that infinitely many colors in P are still left.

We see that whenever f is an embedding of \mathcal{C} -structures then $K(f)$ respects both the ordering and the coloring. To finish the proof we must show that there are no monochromatic triangles in $K(x)$. Assume that there is one $\xi < \psi < \mu$. This means that all pairs are in the same equivalence class. There are isomorphisms i, j which witness that for $\{\xi < \psi\} \sim \{\xi < \mu\}$ and $\{\xi < \mu\} \sim \{\psi < \mu\}$. We know that then i is identity on $\text{supp}(\xi)$ and j is identity on $\text{supp}(\mu)$. Because the triangle is not degenerated, there is $v \in \text{supp}(\psi) \setminus \text{supp}(\mu)$, for such v it holds that $w = i(v) \neq v$ and $j(i(v)) = j(w) = w \neq v$. There must be some $z \in \text{supp}(\xi)$ such that $j(z) = v$. Then we have that $c(\{v, z\}) = c(\{w, z\})$ which is witnessed by i , and $c(\{v, w\}) = c(\{z, w\})$ which is witnessed by j . This is a contradiction, because then $\{v, w, z\}$ is a monochromatic triangle in $x \in \mathcal{C}$. \square

3 Final remarks

Our result suggests the following question: Is there a Fraïssé class without a faithful functor like in Theorem 1?

It can be easily verified that if $\sigma\mathcal{C}$ has push-outs, then for all $x \in \mathcal{C}$, all one-point extensions $\langle x, t \rangle \in \mathcal{C}$ and all embeddings $f: x \rightarrow \text{Flim } \mathcal{C}$ there is a one-point homogenous extension $\langle \text{Flim } \mathcal{C}, t \rangle$ of $\text{Flim } \mathcal{C}$ over $\langle x, t \rangle$. That follows from the fact

that the extension of $\text{Flim } \mathcal{C}$ can be defined to be push out of the following diagram

$$\begin{array}{ccc} \langle x, t \rangle & \xrightarrow{i_0} & \langle \text{Flim } \mathcal{C}, t \rangle \\ i \uparrow & & \uparrow f_0 \\ x & \xrightarrow{f} & \text{Flim } \mathcal{C} \end{array}$$

one can easily verify that it is really an homogeneous one-point extension. This construction works for arbitrary object $X \in \sigma\mathcal{C}$ instead of $\text{Flim } \mathcal{C}$. Let us finally state that the category of locally finite countable groups does not have a push out which brings us to the following concrete open question.

Question 1. Is there a Katětov functor for the Fraïssé class of finite groups? What can we say about one-point homogeneous extensions in this category?

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